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# On the Hermitian Projective Line as a Home for the Geometry of Quantum Theory

Wolfgang Bertram\*

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**Abstract.** In the paper “Is there a Jordan geometry underlying quantum physics?” [Be08], *generalized projective geometries* have been proposed as a framework for a geometric formulation of Quantum Theory. In the present note, we refine this proposition by discussing further structural features of Quantum Theory: the link with *associative involutive algebras*  $\mathbb{A}$  and with *Jordan-Lie* and *Lie-Jordan algebras*. The associated geometries are (*Hermitian*) *projective lines over*  $\mathbb{A}$ ; their axiomatic definition and theory will be given in subsequent work with M. Kinyon [BeKi08].

**AMS subject classification:** primary: 17C37, 17C90, 81R99, secondary: 53Z05, 81P05

**Key words:** generalized projective geometries, Jordan algebras (-triple systems, -pairs), associative geometry, quantum theory

## Introduction

### 0.1 The geometry of quantum theory

The quest for the “Geometry of Quantum Theory” goes back almost to the days when John von Neumann laid the axiomatic foundations of the theory: it seems that von Neumann himself was not entirely satisfied by the non-geometric and linear character of the axiomatic foundations of Quantum Theory – together with G. Birkhoff he tried to base the theory on more fundamental and geometric concepts called the “logic” of quantum theory; the beautiful book “The geometry of quantum theory” [Va85] gives a full account on these and subsequent developments. As a sort of conclusion, the author says (loc. cit., p. 6): “...quantum mechanical systems are those whose logic form some sort of projective geometry”.

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Coming from rather different starting points, various other authors have arrived at similar conclusions: they propose to model the geometry of quantum theory on infinite dimensional manifolds, among which infinite dimensional projective spaces  $\mathbb{P}\mathcal{H}$  (the space of pure states, where  $\mathcal{H}$  is an infinite dimensional Hilbert space) play a central rôle, see, e.g., [Ki79], [AS98], [BH01], [CGM03]. In [CGM03], this approach, named “delinearization program”, is motivated as follows: “The delinearization program, by itself, is not related in our opinion to attempts to construct a non-linear extension of QM with operators that act non-linearly on the Hilbert space  $\mathcal{H}$ . The true aim of the delinearization program is to free the mathematical foundations of QM from any reference to linear structure and to linear operators. It appears very gratifying to be aware of how naturally geometric concepts describe the more relevant aspects of ordinary QM, suggesting that the geometric approach could be very useful also in solving open problems in Quantum Theories.”

In the paper [Be08], the present author proposed an approach following the same general principles, but with the significant difference that we try to geometrize rather the space of *observables* as primary object, and not so much the space of (*pure*) *states*. Remarkably, the resulting geometries still share many features with the projective spaces  $\mathbb{P}\mathcal{H}$ ; we therefore call them *generalized projective geometries*. These form an interesting and quite large category that seems to be suitable for a geometrical formulation of some aspects of quantum theory. However, this framework still is too general – as already pointed out in [Be08], it seems that Nature has chosen among these geometries a fairly special one, namely a geometry that resembles in many respects a projective *line*. Of course, by this we do not mean a usual projective line  $\mathbb{K}\mathbb{P}^1$  over a commutative field  $\mathbb{K}$ , but rather a kind of projective line over an infinite dimensional  $\ast$ -algebra  $\mathbb{A}$ , called the *Hermitian projective line*. In this note, we will present their definition, as well as their mathematical genes which come from the theory of associative and *Jordan* algebras.

## 0.2 Jordan and Lie structures

Our definition of generalized projective geometries has its origin in *Jordan theory*: the associative product  $xy$  in an associative algebra can be decomposed into a symmetric and a skew-symmetric part:

$$xy = \frac{xy + yx}{2} + \frac{xy - yx}{2} =: x \bullet y + \frac{[x, y]}{2}. \quad (0.1)$$

The skew-symmetric part  $[x, y]$  gives rise to a *Lie algebra*, and the symmetric part  $x \bullet y$  to a *Jordan algebra*. Axiomatically, since the foundational work of Pascual Jordan [J32], these algebras are defined by the following two properties (cf. [McC04])

- (J1)  $x \bullet y = y \bullet x$  (commutativity),
- (J2)  $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$  (the Jordan identity).

Not every Jordan algebra is a subalgebra of an associative algebra with respect to the symmetrized product; if this is the case, the Jordan algebra is called *special*. For instance, the Jordan algebra of observables in Quantum Mechanics, the algebra  $\text{Herm}(\mathcal{H})$  of Hermitian operators, clearly is special. In contrast to the case of Lie algebras, it is not easy to get a feeling for the nature of Jordan algebras in an axiomatic approach based on the defining identities (J1) and (J2). Indeed, in the author’s opinion, it is much more appropriate to

start to study some more general objects whose structure is, in a certain sense, much simpler, namely *Jordan triple systems* and *(linear) Jordan pairs*. They are easily interpreted in terms of *3-graded Lie algebras*, see [Be08] or [Be07] for an elementary exposition.

Geometrically, Jordan pairs correspond to generalized projective geometries in a similar way as Lie algebras correspond to Lie groups, and Jordan triple systems correspond to such geometries together with a suitable involution, called *(generalized) polar geometries*. Jordan *algebras* are fairly complicated objects since the corresponding geometries carry all the preceding structures, plus an additional one, called an *absolute null system*. This means, roughly, that the geometry is *canonically* isomorphic to its dual geometry. For instance, among ordinary projective spaces  $\mathbb{K}\mathbb{P}^n$ , only the projective line  $\mathbb{K}\mathbb{P}^1$  has this feature: it is *canonically* isomorphic to its dual projective space of hyperplanes (since only in dimension 1 a hyperplane is a point!). Therefore geometries corresponding to Jordan algebras can be considered as “non-associative generalizations of the projective line” – see [Be08] for all this.

Now, as already mentioned above, the Jordan algebra of Quantum Mechanics,  $\text{Herm}(\mathcal{H})$ , is *special*, and it even is very special in the sense that it is a space of *Hermitian elements in an associative  $\ast$ -algebra*. Therefore there must be some additional geometric structure on the corresponding “projective line”, corresponding to the additional algebraic structure given by the *associative* product: we may call it “associative geometry”. What sort of geometry is this? Surprisingly, it seems that this question has never been seriously investigated. The purpose of the present Note is to give the necessary mathematical definitions and background for its understanding; the “associative geometry” itself will be axiomatically defined and investigated in subsequent work with Michael Kinyon ([BeKi08]).

Algebraically, “associative geometry” in this sense is closely related to concepts of “Jordan-Lie” and “Lie-Jordan” algebras that have appeared in the literature, and which in a sense are axiomatic versions of decompositions, like (0.1), of the associative product. We recall the basic definitions and give some comments on them (Chapter 2); a very remarkable feature is that they introduce a “coupling constant”  $C$  measuring the way in which the Jordan- and Lie-structures are linked to each other. For  $C = 0$ , we essentially get commutative Poisson-algebras, whereas for  $C = 1$  and  $C = -1$  we get two different properly “quantum” structures. The “classical limit”  $C \rightarrow 0$  thus is interpreted as “going from commutative non-associative to commutative associative”. This calls for comparison with the philosophy of Non-commutative Geometry – see the final Chapter 3 for some concluding remarks.

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# 1 Geometry of the Hermitian projective line

## 1.1 The projective line over a ring

Let  $\mathbb{A}$  be an associative algebra, defined over some commutative base field or ring  $\mathbb{K}$ . The *projective line over  $\mathbb{A}$*  is, by definition (cf., e.g., the article by A. Herzer in [Bue95]), the set  $\mathbb{A}\mathbb{P}^1$  of all submodules  $x$  of the right  $\mathbb{A}$ -module  $\mathbb{A} \oplus \mathbb{A}$  that are isomorphic to  $\mathbb{A}$  and admit a complementary submodule  $x'$  isomorphic to  $\mathbb{A}$ . The projective line  $\mathbb{A}\mathbb{P}^1$  is non-empty since it contains at least the two elements

$$o^+ := 0 \oplus \mathbb{A} = e_2 \mathbb{A} \text{ (second factor)}, \quad o^- := \mathbb{A} \oplus 0 = e_1 \mathbb{A} \text{ (first factor)}.$$

The general  $\mathbb{A}$ -linear group  $G := \mathrm{GL}(2, \mathbb{A})$  acts in the usual way on  $\mathbb{A}^2$  from the left; this action permutes all  $\mathbb{A}$ -right modules and defines a transitive action of  $G$  on  $\mathbb{A}\mathbb{P}^1$ : if  $x \in \mathbb{A}\mathbb{P}^1$  with base vector  $v$ , having a complement  $x'$  with base vector  $v'$ , just let  $g$  be the  $\mathbb{A}$ -linear map sending  $e_2$  to  $v$  and  $e_1$  to  $v'$ ; then  $go^+ = x$  (and  $go^- = x'$ ). The stabilizer of  $o^+$ , resp. of  $o^-$ , is the subgroup  $P^-$  of lower (resp. upper) triangular matrices in  $G$ , so that  $\mathbb{A}\mathbb{P}^1$  can be written as a homogeneous space

$$\mathcal{X}^+ := G.o^+ = G/P^-, \quad \mathcal{X}^- := G.o^- = G/P^+.$$

Of course,  $\mathcal{X}^+ = \mathcal{X}^- = \mathbb{A}\mathbb{P}^1$  as sets; but the base points are different.

We say that a pair  $(x, y) \in \mathbb{A}\mathbb{P}^1 \times \mathbb{A}\mathbb{P}^1$  is *transversal* (in incidence geometry one also uses the term *remote*, cf. [Bue95], loc. cit., meaning “as non-incident as possible”), and then write  $x \top y$ , if  $x$  and  $y$  are complementary subspaces:

$$\mathbb{A} \oplus \mathbb{A} = x \oplus y$$

We sometimes write  $(\mathcal{X}^+ \times \mathcal{X}^-)^\top$  for the set of transversal pairs. This set is non-empty since the pair  $(o^+, o^-)$  is transversal, and the action of  $\mathrm{GL}(2, \mathbb{A})$  on it is transitive (just define the matrix  $g$  as above). The stabilizer of the canonical base point  $(o^+, o^-)$  is the subgroup  $H$  of  $G$  consisting of diagonal matrices:

$$(\mathcal{X}^+ \times \mathcal{X}^-)^\top = G.(o^+, o^-) = G/H.$$

The projective line can be seen, in a natural way, as a “projective completion of the algebra  $\mathbb{A}$ ”: let, for any  $y \in \mathbb{A}\mathbb{P}^1$ ,

$$y^\top := \{x \in \mathbb{A}\mathbb{P}^1 \mid x \top y\},$$

the set of elements that are transversal to  $y$ . Then  $y^\top$  is, in a natural way, an affine space over  $\mathbb{K}$  isomorphic to  $\mathbb{A}$ : indeed, since the action of  $\mathrm{GL}(2, \mathbb{A})$  is transitive, without loss of generality we may assume that  $y = o^-$  is the first factor. But all complements of the first factor have a unique base vector of the form  $(a, 1)$  with  $a \in \mathbb{A}$ . In other words, the subgroup of matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in \mathbb{A},$$

acts simply transitively on the set of complements of the first factor. In particular, we may canonically identify  $\mathbb{A}$  with  $(o^+)^\top$  or with  $(o^-)^\top$ , and regard the projective line as some sort of “projective completion” of  $\mathbb{A}$ .

*Examples (cf. [Be07], [Be08], [BeNe05]).*

(1) Assume  $\mathbb{A}$  is a skew-field, or, in other words, an associative division algebra over  $\mathbb{K}$ . Then we can write  $\mathbb{AP}^1 = \mathbb{A} \cup \{\infty\}$ , where  $\infty = o^-$  is the “unique point at infinity of the affine part  $\mathbb{A}$ ”. In particular, if  $\mathbb{A} = \mathbb{R}, \mathbb{C}$  or another locally compact topological field, then  $\mathbb{AP}^1$  is the one-point compactification of  $\mathbb{A}$ .

If  $\mathbb{A}$  is not a skew-field, the “set at infinity” has more than just one element, and its geometric structure is richer and more interesting.

(2) If  $\mathbb{A} = \mathbb{R}[\varepsilon]$ ,  $\varepsilon^2 = 0$ , is the ring of dual numbers over  $\mathbb{R}$ , then  $\mathbb{AP}^1$  is the tangent bundle  $T(\mathbb{RP}^1)$  of the usual real projective line. The set at infinity is the tangent space  $T_\infty(\mathbb{RP}^1)$  at the point  $\infty$  of  $\mathbb{RP}^1$ .

(3) If  $\mathbb{A} = M(n, n; \mathbb{K})$  is the matrix algebra over a commutative unital ring  $\mathbb{K}$ , then  $\mathbb{AP}^1$  is naturally isomorphic to the Grassmannian manifold of  $n$ -spaces in  $\mathbb{K}^{2n}$ .

(4) If  $\mathbb{A} = F(M, \mathbb{K})$  is the commutative algebra of all functions from a set  $M$  to the commutative base ring  $\mathbb{K}$ , then  $\mathbb{AP}^1$  is the space of all functions from  $M$  to the projective line  $\mathbb{KP}^1$ .

(5) If  $\mathbb{A}$  is a principal ideal ring, then  $\mathbb{AP}^1$  is the one-point completion of the quotient field  $\mathbb{F}_\mathbb{A}$  of  $\mathbb{A}$ ; in other words, it is the projective line over the field  $\mathbb{F}_\mathbb{A}$ . Indeed, let  $q \in \mathbb{F}_\mathbb{A}$ . If  $q = 0$ , we associate to it the point  $x = [\text{pr}_1] = o^-$  of  $\mathbb{AP}^1$ . Else write  $q = \frac{s}{r}$  with  $r$  and  $s$  relatively prime in  $\mathbb{A}$ , so there exist  $a, b \in \mathbb{A}$  with  $ar - bs = 1$ , hence the matrix  $g := \begin{pmatrix} s & a \\ r & b \end{pmatrix}$  is invertible. The point  $x := g \cdot [\text{pr}_1] \in \mathbb{AP}^1$  (submodule with base vector  $(s, r)$ ) only depends on  $q$ . Conversely, let  $x \in \mathbb{AP}^1$  generated by  $(s, r) \in \mathbb{A}^2$ . Then the vector  $(s, r)$  can be completed by a vector  $(a, b) \in \mathbb{A}^2$  to a matrix  $g$  as above with determinant equal to 1. If  $s = 0$ , then  $x = [\text{pr}_1]$ ; if  $r = 0$ , then  $x = [\text{pr}_2]$ , and in all other cases  $x$  can be identified with the element  $q = \frac{s}{r}$  of the quotient field. Both constructions are inverse to each other, and thus we have a bijection between  $\mathbb{F}_\mathbb{A} \cup \{\infty\}$  and  $\mathbb{AP}^1$ .

For instance,  $\mathbb{ZP}^1$  is the rational projective line  $\mathbb{QP}^1$ , and if  $\mathbb{A} = \mathbb{K}[X]$  is the polynomial ring over a field  $\mathbb{K}$ , then  $\mathbb{AP}^1 = \mathbb{K}(X) \cup \{\infty\}$  is the completion of the rational function field by a “function”  $\infty$  which can be considered as the inverse of the zero function. In both cases, the “set at infinity” is very big: it is a sort of infinite dimensional manifold over  $\mathbb{A}$ .

(6) Let us assume that  $\mathbb{A}$  is a *continuous inverse algebra (c.i.a.)*: a topological algebra over a topological ring  $\mathbb{K}$  such that the unit group  $\mathbb{A}^\times$  is open in  $\mathbb{A}$  and inversion  $i : \mathbb{A}^\times \rightarrow \mathbb{A}$  is a continuous map. We assume also that  $\mathbb{K}^\times$  is dense in  $\mathbb{K}$  (in particular, the topology is not discrete). Then inversion is actually smooth over  $\mathbb{K}$ , and  $\mathbb{AP}^1$  is a smooth manifold over  $\mathbb{K}$ , modelled on the topological linear space  $\mathbb{A}$  (see [BeNe05]). For instance,  $\mathbb{A}$  may be any Banach algebra over  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{Q}_p$ .

## 1.2 The Hermitian projective line

Next we consider an associative algebra  $\mathbb{A}$  with an *involution*  $*$  :  $\mathbb{A} \rightarrow \mathbb{A}$ ,  $a \mapsto a^*$  (antiautomorphism of order 2 stabilizing  $\mathbb{K} = \mathbb{K}1$ ; following standard terminology [Le06] we say that  $*$  is *of the first kind* if  $*$  induces the identity on  $\mathbb{K}$ , and *of the second kind* else). Then the involution  $*$  lifts to an involution of the projective line  $\mathbb{A}\mathbb{P}^1$  whose fixed point set is called the *Hermitian projective line*, see [BeNe05]. Let us give here a slightly modified version of the construction given in loc. cit.: for any matrix  $B = (b_{ij}) \in M(2, 2; \mathbb{A})$  we may define a sesquilinear form  $\beta = \beta_B$  on  $\mathbb{A}^2$  by

$$\beta(x, y) = \sum_{i,j=1}^2 x_i^* b_{ij} y_j.$$

The sesquilinearity property reads

$$\forall \lambda, \mu \in \mathbb{A} : \quad \beta(x\lambda, y\mu) = \lambda^* \beta(x, y) \mu,$$

hence the orthogonal complement  $E^{\perp, \beta}$  of an  $\mathbb{A}$ -right submodule  $E$  is again an  $\mathbb{A}$ -right submodule. Assume now that  $\beta$  is *non-degenerate*, i.e.,  $B$  is an invertible matrix. Then, if  $E \in \mathbb{A}\mathbb{P}^1$ , also  $E^{\perp, \beta} \in \mathbb{A}\mathbb{P}^1$ : indeed, if  $\mathbb{A} \oplus \mathbb{A} = x \oplus y$  is a direct sum decomposition with both factors  $x$  and  $y$  isomorphic to  $\mathbb{A}$ , then so is  $\mathbb{A} \oplus \mathbb{A} = x^\perp \oplus y^\perp$ . Moreover, the map thus defined

$$\perp^\beta : \mathbb{A}\mathbb{P}^1 \rightarrow \mathbb{A}\mathbb{P}^1, \quad E \mapsto E^{\perp, \beta}.$$

is a bijection (with inverse corresponding to  $B^{-1}$ ). It is  $G$ -equivariant in the following sense:  $(g.x)^\perp = \phi(g)^{-1}.x^\perp$  where  $\phi$  is the anti-automorphism “ $\beta$ -adjoint” of  $M(2, 2; \mathbb{A})$  given by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B^{-1} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} B.$$

Now let us consider the following three sesquilinear forms on  $\mathbb{A}^2$  given by

$$\begin{aligned} \omega((x_1, x_2), (y_1, y_2)) &= x_1^* y_2 - x_2^* y_1, \\ \vartheta((x_1, x_2), (y_1, y_2)) &= x_1^* y_2 + x_2^* y_1, \\ \sigma((x_1, x_2), (y_1, y_2)) &= x_1^* y_1 - x_2^* y_2, \end{aligned}$$

corresponding to the three matrices

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S := I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

called the “\*-symplectic”, “\*-hyperbolic”, and “\*-symmetric” forms. Since these forms are Hermitian, resp. skew-Hermitian, the corresponding maps  $\perp^\beta$  are involutions, i.e., of order 2. The fixed point sets

$$\begin{aligned} \mathbb{P}_h &:= (\mathbb{A}\mathbb{P}^1)^{\perp, \omega} = \{E \in \mathbb{A}\mathbb{P}^1 \mid E^{\perp, \omega} = E\}, \\ \mathbb{P}_{sh} &:= (\mathbb{A}\mathbb{P}^1)^{\perp, \vartheta} = \{E \in \mathbb{A}\mathbb{P}^1 \mid E^{\perp, \vartheta} = E\}, \\ \mathbb{P}_u &:= (\mathbb{A}\mathbb{P}^1)^{\perp, \sigma} = \{E \in \mathbb{A}\mathbb{P}^1 \mid E^{\perp, \sigma} = E\} \end{aligned}$$

are called the *Hermitian*, respectively *skew-Hermitian* and *unitary projective line over the involutive algebra*  $(A, *)$ . To justify the last terminology, note that the *\*-unitary group*

$$U(\mathbb{A}, *) := \{a \in \mathbb{A} \mid a^*a = aa^* = 1\}$$

is imbedded into  $\mathbb{P}_u$  via  $a \mapsto (1, a)\mathbb{A}$  (to see this, just note  $\sigma((1, a), (1, a)) = 1 - a^*a = 0$ ). In some cases, this imbedding is a bijection (see examples below). Of course, via a base change the forms  $\vartheta$  and  $\sigma$  are isomorphic, and therefore also the skew-Hermitian and the unitary projective line are isomorphic. In general, they are not isomorphic to the Hermitian projective line; but in some interesting special cases they are (see below).

Let us denote by  $\mathcal{X}$  one of the three kinds of projective line  $\mathbb{P}_h, \mathbb{P}_{sh}, \mathbb{P}_u$  just defined. There are some general features already encountered for the full projective line  $\mathbb{A}\mathbb{P}^1$  that carry over: the notion of *transversality in*  $\mathcal{X}$  remains the same; we have a transversal pair of base points: for  $\mathbb{P}_h$  and  $\mathbb{P}_{sh}$  it is again  $(o^+, o^-)$ ; for  $\mathbb{P}_u$  we rather have to take the two diagonals in  $\mathbb{A} \oplus \mathbb{A}$ . There is a natural group  $G$  acting, namely the unitary groups of the respective forms. For  $\omega$ , we call it the *\*-symplectic group*, denoted by  $\text{Sp}(\mathbb{A}, \omega)$ ; for  $\vartheta$ , the *\*-pseudo unitary group*, denoted by  $U(\mathbb{A}, \mathbb{A}, *)$ . One can show that, for  $\omega$ , this action is always transitive both on the Hermitian projective line and on the set of transversal pairs, so we may consider homogeneous spaces of the form  $G/P^-, G/P^+, G/H$  as before, whereas for  $\vartheta$ , this need not always be the case: in this case we better write

$$\mathcal{X}^+ := U(\mathbb{A}, \mathbb{A}, *) \cdot o^+, \quad \mathcal{X}^- := U(\mathbb{A}, \mathbb{A}, *) \cdot o^-,$$

and these two orbits may be equal or disjoint in the skew-Hermitian projective line. In any case, it remains true that the set  $y^\top$  of all transversal elements in  $\mathcal{X}$  to a given element  $y$  is always an affine space over  $\mathbb{K}$ , which now is modelled on the sets

$$\text{Herm}(\mathbb{A}, *) := \{a \in \mathbb{A} \mid a^* = a\}, \quad \text{respectively} \quad \text{Aherm}(\mathbb{A}, *) := \{a \in \mathbb{A} \mid a^* = -a\}$$

for the Hermitian, resp. skew-Hermitian, projective line.

*Examples (cf. [Be07], [Be08], [BeNe05]).*

(1) If  $\mathbb{A}$  is commutative, then  $\text{Herm}(\mathbb{A}, *)$  is also a commutative algebra, and  $\mathbb{P}_h$  is just the projective line over this algebra. If  $\mathbb{A} = F(M, \mathbb{K})$  is a function algebra, then  $\mathbb{P}_u$  is the group of functions with values in the “circle group”  $\{r \in \mathbb{K}^\times \mid r^* = r^{-1}\}$ .

(2) If  $\mathbb{A} = M(n, n; \mathbb{K})$  and  $X^* = X^t$ , then  $\mathbb{P}_h$  is the variety of Lagrangian subspaces of the canonical symplectic form on  $\mathbb{K}^{2n}$ , and  $\mathbb{P}_{sh}$  is the Lagrangian variety for the quadratic form of signature  $(n, n)$ . For  $\mathbb{K} = \mathbb{R}$ , the imbedding of the orthogonal group  $O(n)$  into the latter is a bijection.

### 1.3 The unitary-Hermitian projective line

Quantum mechanics requires to work over the field  $\mathbb{C}$  of complex numbers and the involution  $*$  to be  $\mathbb{C}$ -antilinear. This has the particular consequence that the spaces of Hermitian elements ( $a^* = a$ ) and skew-Hermitian elements ( $a^* = -a$ ) are isomorphic. In general, let us call an involution  $*$  of an associative  $\mathbb{K}$ -algebra  $\mathbb{A}$  of *complex type* if there exists



an element  $i \in \mathbb{K}$  with  $i^2 = -1$  and  $i^* = -i$ . Then multiplication by  $i$  is a  $\mathbb{K}$ -linear isomorphism from  $\text{Herm}(\mathbb{A}, *)$  onto  $\text{Aherm}(\mathbb{A}, *)$ . The diagonal matrix  $\text{dia}(i, 1) \in \text{Gl}(2, \mathbb{A})$  then induces a bijection from the Hermitian projective line onto the skew-Hermitian one. Mathematically, this is a very special feature; but nevertheless Nature has chosen it as being part of the structure of Quantum Mechanics:

*Example.* Let  $\mathbb{A} = M(n, n; \mathbb{C})$  and  $X^* = \overline{X}^t$ . Then  $\mathbb{P}_h$ ,  $\mathbb{P}_{sh}$  and  $\mathbb{P}_u$  are all isomorphic to the variety of all Lagrangian subspaces of the Hermitian form on  $\mathbb{C}^{2n}$  with signature  $(n, n)$ . The imbedding of the unitary group  $U(\mathbb{A}; *) = U(n)$  into  $\mathbb{P}_u$  is in this case a bijection. The “big” group  $G$  acting on  $\mathbb{P}_h \cong U(n)$  is  $\text{PU}(n, n)$ . To get the setting of Quantum Mechanics, one may in this example replace  $M(n, n; \mathbb{C})$  by the bounded operators on an infinite dimensional Hilbert space  $\mathcal{H}$  (unbounded operators can be dealt with by a more subtle choice of *Jordan pair*, see below).

One may also replace the positive involution considered above by the “indefinite involution”  $X^* := I_{p,q} \overline{X}^t I_{p,q}$ , where  $I_{p,q}$  is the usual diagonal matrix of signature  $(p, q)$  and square one. As spaces,  $\mathbb{P}_h$ ,  $\mathbb{P}_{sh}$  and  $\mathbb{P}_u$  are then still the same as above, the only thing that changes is that we now consider another “polarity”, which corresponds to the imbedding of the pseudo-unitary group  $U(\mathbb{A}, *) = U(p, q)$  as open dense set into the compact space  $\mathbb{P}_u$  (in the Russian literature this is called the “Potapov-Ginzburg transformation”).

## 1.4 Positivity

There is another particular feature of Quantum Mechanics that one has to take account of: *positivity*. Algebraically, this corresponds to the positivity condition in the definition of a  $C^*$ -algebra ( $\|xx^*\| = \|x\|^2$ ); geometrically, it corresponds to the fact that there exists a partial order on the Jordan algebra  $\text{Herm}(\mathbb{A}, *)$  such that squares are positive (in particular,  $\text{Herm}(\mathbb{K}, *)$  then has to be an ordered ring or field), and also to the fact that (under some additional conditions) the imbedding of the unitary group  $U(\mathbb{A}, *)$  into  $\mathbb{P}_u$  becomes a bijection. These are interesting topics for further work; however, for the logical development of the theory it seems useful not to introduce such positivity assumptions at an early stage.

# 2 Jordan-, Lie- and Jordan-Lie algebras

## 2.1 Jordan pairs and triple systems

For the sake of completeness, we just recall here the mere definitions: a (*linear*) *Jordan pair* is a pair  $(V^+, V^-)$  of  $\mathbb{K}$ -modules together with two trilinear maps  $T^\pm : V^\pm \times V^\mp \times V^\pm \rightarrow V^\pm$  satisfying the following identities (LJP1) and (LJP2):

$$\text{(LJP1)} \quad T^\pm(x, y, z) = T^\pm(z, y, x),$$

$$\text{(LJP2)} \quad T^\pm(a, b, T^\pm(x, y, z)) =$$

$$T^\pm(T^\pm(a, b, x), y, z) - T^\pm(x, T^\mp(b, a, y), z) + T^\pm(x, y, T^\pm(a, b, z)).$$

The basic example of a linear Jordan pair is given by spaces of rectangular matrices:

$$(V^+, V^-) = (\text{Hom}(F, E), \text{Hom}(E, F)) \quad \text{with} \quad T^\pm(x, y, z) = xyz + zyx.$$

By definition, a *Jordan triple system* (JTS) is a  $\mathbb{K}$ -module  $V$  together with a trilinear map  $T : V \times V \times V \rightarrow V$  satisfying the identities (JT1) and (JT2) obtained from (LJP1) and (LJP2) by omitting the superscripts  $\pm$ . The basic example is again the space of rectangular matrices,

$$V = M(p, q; \mathbb{K}) \quad \text{with} \quad T(x, y, z) = xy^t z + zy^t x. \quad (2.1)$$

Note that the transpose corresponds to the choice of a scalar product. If  $p = q$ , or if  $V = \mathbb{A}$  is any associative algebra, one also has another Jordan triple product given by  $T(a, b, c) = abc + cba$ . In general, every JTS gives rise to a Jordan pair  $(V^+, V^-) := (V, V)$  with  $T^+ = T^- = T$  (but not every Jordan pair is of this form), and every Jordan algebra (with product  $\bullet$ ) gives rise to a JTS via

$$T(x, y, z) = \frac{1}{2}(x \bullet (y \bullet z) - y \bullet (x \bullet z) + (x \bullet y) \bullet z). \quad (2.2)$$

Summing up, there are several functors between the following categories: associative algebras; Jordan algebras; Jordan triple systems; Jordan pairs (and several others, such as: Lie triple systems; associative and alternative pairs, etc). Each of these functors sheds light on certain features of the categories between which it is defined.

## 2.2 Jordan-Lie algebras

As already noticed above, some algebras carry simultaneously the structure of a Jordan and of a Lie algebra: on the one hand, we have the full associative algebras  $\mathbb{A}$  with the usual (anti-) commutators; on the other hand, the spaces  $\text{Herm}(\mathcal{H})$  of Hermitian operators in a complex Hilbert space  $\mathcal{H}$ , where a factor  $i$  comes in. The concept of a *Jordan-Lie algebra* takes account of both cases; the definition is due to G. Emch ([E84]), although the concept seems to have appeared first in the paper [GP76].

**Definition.** Let  $C \in \mathbb{K}$  be a constant. A  $\mathbb{K}$ -module  $V$ , equipped with two bilinear products  $[x, y]$  and  $x \bullet y$  is called a *Jordan-Lie algebra (with coupling constant  $C$ )* if

(JL1)  $(V, [\cdot, \cdot])$  is a Lie algebra;

(JL2)  $(V, \bullet)$  is a Jordan algebra;

(JL3) the Lie algebra acts by derivations of the Jordan algebra, that is,

$$[x, u \bullet v] = [x, u] \bullet v + u \bullet [x, v],$$

(JL4) the associators of both products are proportional:

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = -C'([x, y], z) - [x, [y, z]].$$

Of course, (JL4) can also be written  $(x \bullet y) \bullet z - x \bullet (y \bullet z) = C[[z, x], y]$ , thanks to the Jacobi identity. The main examples are:

- (1) Commutative Poisson algebras: for  $C = 0$ , Condition (JL4) says that  $\bullet$  is a commutative and associative product, on which the Lie algebra acts by derivations, by (JL3).
- (2) Associative algebras  $V = \mathbb{A}$  with usual Jordan product and Lie bracket: (JL3) is clear since the Lie algebra already derives the associative product; (JL4) with  $C = 4$  follows by a direct calculation.
- (3) Hermitian elements: under the assumptions from Section 1.3, let  $V = \text{Herm}(\mathbb{A}, *)$  with its usual Jordan product and the modified Lie bracket  $[x, y] := i(xy - yx)$ . The same calculation as in the preceding example yields an additional factor  $i^2 = -1$  on the right hand side of (JL4), whence we get a Jordan-Lie algebra with coupling constant  $C = -4$ .

Conversely, given a Jordan-Lie algebra with coupling constant  $C$ , consider the scalar extension  $R := \mathbb{K}[X]/(X^2 - C)$  of  $\mathbb{K}$ ; writing  $i := [X]$ , this is simply the ring  $\mathbb{K} \oplus i\mathbb{K}$  with defining relation  $i^2 = C$ . (For  $C = 0$ , these are the dual numbers over  $\mathbb{K}$ .) Let  $V_R := V \oplus iV$  the scalar extension of the given Jordan-Lie algebra, which is again a Jordan-Lie algebra, now defined over  $R$ . Define a new product on  $V_R$  by

$$xy := x \bullet y + i[x, y].$$

By a direct calculation (cf. [E84], p. 307) one sees that the associator of this product is

$$(xy)z - x(yz) = C[[x, z], y] - i^2([x, y], z) - [[y, z], x] = (C - i^2)[[x, z], y] = 0,$$

hence  $V_R$  is an associative algebra. The “conjugation map”  $a + ib \mapsto a - ib$  is an involution of this algebra. If  $C$  is a square in  $\mathbb{K}^\times$ , we get back Example (2), and if  $-C$  is a square in  $\mathbb{K}^\times$ , we get back Example (3). If  $C = 0$ , we have constructed an associative, in general non-commutative algebra out of a commutative Poisson algebra; it is a sort of first approximation of a deformation quantization of that algebra.

Categorical notions for Jordan-Lie algebras follow the usual pattern: homomorphisms, ideals, simple, semi-simple objects are those which have the corresponding properties both for the Jordan and the Lie product; for invertible  $C$ , simple objects then correspond to simple associative algebras over  $R$  with involution. For  $\mathbb{K} = \mathbb{R}$ , the classification of simple finite-dimensional objects is therefore very easy: for  $C > 0$ , we get simple associative algebras (that is, matrix algebras over the three associative real division algebras, by the classical Burnside theorem); for  $C < 0$ , we have to look at simple complex algebras  $M(n, n; \mathbb{C})$  with  $\mathbb{C}$ -antilinear involution: it is known that all such involutions correspond to adjoints with respect to a non-degenerate Hermitian form on  $\mathbb{C}^n$  (see [Le06]). Therefore the Jordan part of a simple finite-dimensional real Jordan-Lie algebra is isomorphic to  $V = \text{Herm}(p, q; \mathbb{C})$  and its Lie part isomorphic to  $\mathfrak{u}(p, q)$ , the Lie algebra of the pseudo-unitary group  $U(p, q)$ . In infinite dimension, the classification of Jordan-Lie algebras contains the classification of  $C^*$ -algebras as a subproblem.

We add two remarks on important special features of Jordan-Lie algebras:

1. Jordan-Lie algebras permit to single out *complex* Hermitian matrices by a *purely real* concept. This means that an axiomatic approach to quantum mechanics without making use of complex numbers is possible. See also [La93] and [La96].

**2.** *Tensor products* exist in the category of Jordan-Lie algebras. This is very remarkable since tensor products neither exist in the category of Jordan algebras nor in the one of Lie algebras. This observation is the starting point of the paper [GP76], where the notion of *composition class* as a class of two-product algebras closed under tensor products is introduced; the idea to characterize quantum and classical mechanics as certain composition classes goes back to Niels Bohr.

## 2.3 Lie-Jordan algebras

The following definition is (for  $C = 1$ ) due to Grishkov and Shestakov [GS01]: Let  $C \in \mathbb{K}$  be a constant. A  $\mathbb{K}$ -module  $V$ , equipped with a bilinear product  $[x, y]$  and a trilinear product  $T : V^3 \rightarrow V$  is called a *Lie-Jordan algebra (with coupling constant  $C$ )* if

(LJ1)  $(V, [\cdot, \cdot])$  is a Lie algebra,

(LJ2)  $(V, T)$  is a JTS,

(LJ3) the Lie algebra acts by derivations of the JTS  $T$ , that is,

$$[x, T(u, v, w)] = T([x, u], v, w) + T(u, [x, v], w) + T(u, v, [x, w]),$$

(LJ4) skew-symmetrized  $T$  is proportional to the triple Lie bracket:

$$T(x, y, z) - T(y, x, z) = C[[x, y], z].$$

Similar comments as in the preceding section can be made. Every Jordan-Lie algebra gives rise to a Lie-Jordan algebra, but the converse is false: the  $-1$ -eigenspace of any involution of an associative algebra  $(\mathbb{A}, *)$  gives rise to a Lie-Jordan algebra (and Grishkov and Shestakov show that every Lie-Jordan algebra is obtained in this way, if  $C = 1$ ). For instance, if  $\mathbb{A} = M(n, n; \mathbb{R})$  and  $X^* = X^t$ , the  $-1$ -eigenspace is the Lie algebra  $\mathfrak{o}(n)$  of the orthogonal group, which is stable under the usual Jordan triple product (2.1), giving rise to Lie-Jordan algebra; but it is not a Jordan-Lie algebra since (2.1) never is obtained from a Jordan algebra via (2.2).

## 3 Comments and afterthoughts

**3.1.** Jordan-Lie algebras (with a certain positivity condition, see [E84]) are equivalent to  $C^*$ -algebras. Therefore there is no contradiction between associative or non-associative geometry in our sense and the philosophy of Non-commutative Geometry: it is just a difference of language. Jordan geometry leads back to a classical language for describing non-classical results; we re-introduce the language of point-spaces on a level where Non-commutative Geometry teaches us to abandon point-spaces. However, the point of view of Jordan-Lie algebras separates the Jordan and Lie aspects of a  $C^*$ -algebra, and thus sheds light onto aspects that remain unnoticed in a purely associative theory.

**3.2.** We think that Bohr's idea of "composition classes" (see Remark 2 in Section 2.2) is interesting and deserves a careful re-investigation. Composition classes with  $C = 0$  are

called “classical” and those with  $C$  invertible are called “quantal”. Classical composition classes are thus commutative Poisson algebras; quantal composition classes are generally non-associative, but also contain associative commutative Jordan algebras (which, however, are then just considered as commutative and not as Poisson algebras: if the left-hand side of (JL4) is zero and  $C$  is invertible, it follows that  $[[V, V], V] = 0$ ). In [GP76] (Appendix A) also the case of symmetric and exterior tensor powers is considered; only the symmetric ones give rise to a composition class. Maybe the extension of such concepts to ternary products (Lie-Jordan algebras) permits to include exterior powers in this picture.

**3.3.** In his work [Pen89], [Pen05], Roger Penrose frequently expresses the opinion that foundational problems of unifying quantum theory with general relativity are related to the coexistence of two different modes of time evolution in quantum mechanics, the “unitary Schrödinger evolution”  $\mathbf{U}$  and the “state reduction”  $\mathbf{R}$ . Mathematically, as already noticed by G. Emch [E84], the “Jordan part” of an observable somehow reflects its “quantum aspect”, that is,  $\mathbf{R}$ , whereas its “Lie part” rather represents  $\mathbf{U}$ . Since the concept of a Jordan-Lie algebra clearly separates these two aspects, it seems to meet the heart of the problem as formulated by Penrose. This indicates that the “coupling axiom” (JL4) is of central importance and should be better understood. In particular, it is not at all easy to find a geometric interpretation of this axiom on the level of the geometries of the corresponding projective lines.

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